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CONDITIONS FOR CRITICALITY IN CERTAIN TYPES OF NUCLEAR REACTORS

By
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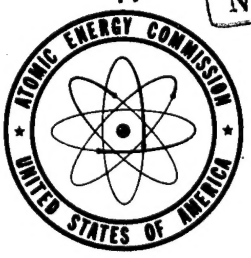

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ABSTRACT

The integro-differential equation describing the diffusion of neutrons in a multiplying medium is studied under the assumption of plane geometry and linearly anisotropic scattering, but without the usual limitation to continuous slowing down and weak absorption. Under the condition that the integrated flux vanish at the boundary, it is shown that a separable solution exists. The relation between core size and the number of neutrons required per fission for a steady state is determined. The results are specialized to the case of isotropic scattering in the laboratory system, and to the case of a medium containing hydrogen and an element which scatters without energy loss. (Continued abstract)

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TABLE OF CONTENTS

Section		Page
I.	Introduction - - - - - - - - - - - - - - -	6
II.	Asymptotic Solution of the Boltzmann Equation -	7
III.	Isotropic Scattering - - - - - - - - - - -	20
IV.	Moderation by Hydrogen - - - - - - - - - -	29
Bibliography - - - - - - - - - - - - - - - - -		34

I. INTRODUCTION

In designing a nuclear reactor, one of the major considerations is the amount of fissionable material required to sustain a chain reaction. If neutrons are moderated by elements of intermediate atomic weight for which inelastic scattering is negligible, and if little absorption is present, the problem can be treated by age - diffusion theory. Two cases of interest, however, do not fall under this formalism: these are moderation by a heavy element, where energy losses are caused primarily by inelastic scattering, and moderation by hydrogen, in which cross-sections change radically in a single collision interval.

The purpose of this report is to study these two cases in terms of the integro-differential equation underlying diffusion theory. The discussion will be limited by the following assumptions: First, the reactor has plane symmetry, so that the neutron distribution depends only upon a single space coordinate. Second, the probability that a neutron will be scattered through a given angle can be represented by a constant plus a term proportional to the cosine of the angle. Finally, the neutron density at the boundaries is zero for all energies.

II. ASYMPTOTIC SOLUTION OF THE BOLTZMANN EQUATION

We assume the reactor consists of a slab bounded by vacuum, extending along the x-axis from -a to +a and indefinitely in the y-z plane. Its time - independent state is then described by the function $N(x,v,\mu)$ dx dv $d\mu$, defined as the number of neutrons in a volume element with unit cross-section and thickness dx, having scalar velocities between v and v + dv, and with velocity vectors making an angle with the positive x-axis whose cosine lies between μ and $\mu + d\mu$. It is slightly more convenient, however, to take as a distribution function

$$F(x,v,\mu) = N(x,v,\mu)\, v \ .$$

Two quantities needed later can be defined in terms of F; these are the flux

$$F_0(x,v) = \int_{-1}^{1} d\mu\, F(x,v,\mu) \tag{2.1}$$

and the current

$$F_1(x,v) = \int_{-1}^{1} d\mu\, \mu\, F(x,v,\mu) \tag{2.2}$$

The form of the continuity equation we shall use requires that the differential scattering cross-section be linear in the scattering angle. The differential scattering cross-section $\sigma_S(v,v',\mu_0)$ dv $d\mu_0$ is defined as the macroscopic cross-section for scattering from the scalar velocity v into the interval from v to v + dv, through an angle (measured in the laboratory system of coordinates) whose cosine lies between μ_0 and $\mu_0 + d\mu_0$. In general $\sigma_s(v,v',\mu_0)$ can be expanded in an infinite series of Legendre polynominals in μ_0. We will make the assumption that only the first two terms are different from zero, and write

$$\sigma_S(v,v',\mu_0) = \sigma_S(v,v')\left[\frac{1}{2} + \frac{3}{2}\bar{\mu}_0(v)\mu_0\right]$$

Using the orthogoality of the Legendre polynominals, and integrating this expression with respect to μ_0, it follows immediately that

$$\sigma_s(v, v') = \int_{-1}^{1} d\mu_0 \, \sigma_s(v, v', \mu_0)$$

and is the cross-section for scattering from v to a unit interval about v through any angle, while

$$\overline{\mu_0}(v) = \frac{\int_{-1}^{1} d\mu_0 \mu_0 \, \sigma_s(v, v', \mu_0)}{\int_{-1}^{1} d\mu_0 \, \sigma_s(v, v', \mu_0)}$$

and is the average cosine of the angle through which a neutron is scattered in a single collision.

Under the assumption that neutron-neutron interactions are negligible the distribution function $F(x, v, \mu)$ satisfies the following equation of continuity due to Boltzmann:

$$\mu \frac{\partial F(x, v, \mu)}{\partial x} + \sigma(v) F(x, v, \mu) = \frac{1}{2} \nu \chi(v) \int_0^{v_0} dv' \sigma_f(v') \int_{-1}^{1} d\mu' F(x, v', \mu') + \frac{1}{2} \int_v^{v_0} dv' \sigma_s(v, v') \int_{-1}^{1} d\mu' F(x, v', \mu') + \frac{3}{2} \mu \int_v^{v_0} dv' \overline{\mu_0}(v') \sigma_s(v, v') \int_{-1}^{1} d\mu' \mu' F(x, v', \mu') \tag{2.3}$$

where $\sigma(v)$ and $\sigma_f(v)$ are the macroscopic total and fission cross-sections respectively for neutrons of velocity v; $\chi(v)dv$ is the fraction of fission neutrons born with velocities between v and $v + dv$, v_0 is some finite velocity above which no fission neutrons are born, and ν is the number of neutrons produced per fission. It is assumed that both ν and χ are independent of the energy of the neutrons causing fission, and that the angular distribution of fission neutrons is isotropic in the laboratory system.

A detailed derivation of equation (2.3) is given by Weinberg[1] whose notation is largely followed here. The two terms on the left

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[1] A. M. Weinberg, Pile Neutron Physics, M-3336, equation 1-28.

represent the loss of neutrons from a unit volume due to leakage through the boundaries and to absorption and scattering within it. The first term on the right corresponds to neutrons produced by fission and the second to neutrons produced by scattering down from a higher energy. The last term represents a distortion in the angular distribution of these scattered neutrons due to the anisotropic scattering in the laboratory system; if $\mu_0 = 0$, this term is absent.

The cross-sections σ, σ_f and σ_s, through which the dependence of F on the material properties of the reactor enters, have been written as dependent only on velocity. However, spatial dependence can be taken into account provided it is of the form

$$\sigma(x,v) = \rho(x)\,\sigma(v)$$

where σ (x,v) refers to any of the cross-sections and $\rho(x)$ is some fixed function of X. If the new coordinate[2]

$$x'(x) = \int_0^x \rho(\xi)\,d\xi$$

is introduced, together with the function

$$F'(x'(x), v, \mu) = F(x, v, \mu)$$

$\rho(x)$ can be cancelled, giving an equation in F' formally identical with (2.3). It is therefore sufficient to consider (2.3) as it stands.

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[2] This transformation is a special case of the optical distance used by B. Davison, Transport Theory of Neutrons, LT-18, Chalk River, Ontario, National Research Council of Canada, January 1947, p. 10.

The problem we consider is the following: knowing the values of the cross-sections and of ν for a particular composition, we require the critical size of the reactor, that is, the size for which a steady state will be maintained. Physical considerations indicate that a solution does not always exist, since enough parasitic absorption may be present to prevent a self-sustaining chain reaction under any conditions. It is therefore more convenient to consider the size as given, and to determine the value of ν required for criticality. We accordingly take the boundaries at $x = \pm a$ as given and look for a solution of (2.3) for $F(x,v,\mu)$ together with the corresponding value of ν, subject to the conditions

$$F(x,v,\mu) \equiv F(-x,v,-\mu) \tag{2.4}$$

$$F_0(x,v) > 0 \quad \text{for} \quad -a < x < a \tag{2.5}$$

$$F_0(a,v) \equiv F_0(-a,v) \equiv 0 \quad \text{for all } v \tag{2.6}$$

The first condition is due to the symmetry about $x = 0$, and the second follows from the physical requirement of a positive neutron density within the reactor. The last condition is only an approximation to the physical system and is introduced (as is usually done when considering energy dependence) to simplify the problem. The proper boundary conditions for a reactor bounded by vacuum are

$$F(a,v,\mu) \equiv 0 \quad \text{for} \quad -1 < \mu < 0$$

$$F(-a,v,\mu) \equiv 0 \quad \text{for} \quad 0 < \mu < 1,$$

corresponding to no incoming neutrons. Under these conditions, the space distribution of the flux has been determined for the case of monoenergetic neutrons (ref. 3), and is found to approach asymptotically the distribution obtained using the approximate condition (2.6),

the difference between the two distributions decreasing away from the boundary like an exponential with a relaxation length of a mean free path. It seems reasonable to expect a similar behavior for the energy dependent case; the error in critical size caused by using the approximate boundary condition should then be small if the reactor thickness is large compared to a mean free path, as it usually is.

In the remainder of this section we will show, following a procedure used by Weinberg[3] for the case of monoenergetic neutrons, that solutions of the energy dependent Boltzmann equation exist having a sinusoidal dependence on x. Since the equation is linear, it is convenient to work with the complex function

$$\mathcal{F}(x,v,\mu) = e^{i\kappa x} f(v,\mu) \tag{2.7}$$

whose real and imaginary parts will both satisfy (2.3) if F itself does. We make the tentative assumption that a solution of the form (2.7) exists and try to determine the function $f(v,\mu)$ and the parameter κ. Substituting this in place of F in (2.3) and cancelling the exponential leads to

$$(\sigma + i\kappa\mu) f(v,\mu) = \tfrac{1}{2} \nu \chi(v) \int_0^{v_0} dv' \sigma_f(v') \int_{-1}^{1} d\mu' f(v',\mu') + \tfrac{1}{2} \int_v^{v_0} dv' \sigma_s(v,v') \int_{-1}^{1} d\mu' f(v',\mu') + \tfrac{3}{2} \mu \int_v^{v_0} dv' \bar{\mu}_0(v') \sigma_s(v,v') \int_{-1}^{1} d\mu' \mu' f(v',\mu') \tag{2.8}$$

Using the following abbreviations,

$$f_0(v) \equiv \int_{-1}^{1} d\mu\, f(v,\mu) \tag{2.9}$$

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[3] Weinberg, op. cit., p. 33 - 38.

$$f_1(v) \equiv i \int_{-1}^{1} d\mu\, \mu\, f(v,\mu) \tag{2.10}$$

$$n(v) \equiv \nu\, \chi(v) \int_0^{V_0} dv'\, \sigma_f(v')\, f_0(v') + \int_v^{V_0} dv'\, \sigma_s(v,v')\, f_0(v') \tag{2.11}$$

$$m(v) \equiv 3 \int_v^{V_0} dv'\, \bar{\mu}_0(v')\, \sigma_s(v,v')\, f_1(v') \tag{2.12}$$

equation (2.8) can be written

$$f(v,\mu) = \tfrac{1}{2} \frac{n(v) - i\mu\, m(v)}{\sigma(v) + i K \mu}. \tag{2.13}$$

Substituting this expression for $f(v,\mu)$ in (2.9) and (2.10) and using the integrals

$$\begin{aligned} \int_{-1}^{1} \frac{d\mu}{1+i\alpha\mu} &= 2\,\frac{\tan^{-1}\alpha}{\alpha} = 2\beta \\ \int_{-1}^{1} \frac{\mu\, d\mu}{1+i\alpha\mu} &= \frac{2}{i\alpha}\left(1 - \frac{\tan^{-1}\alpha}{\alpha}\right) = \frac{2}{i}\,\frac{1-\beta}{\alpha} \\ \int_{-1}^{1} \frac{\mu^2\, d\mu}{1+i\alpha\mu} &= \frac{2}{\alpha^2}\left(1 - \frac{\tan^{-1}\alpha}{\alpha}\right) = 2\,\frac{1-\beta}{\alpha^2} \end{aligned} \tag{2.14}$$

where $\alpha = K\sigma$ and $\beta = \tan^{-1}\alpha / \alpha$, we have

$$f_0(v) = \frac{1}{\sigma}\left(\beta n - \frac{1-\beta}{\alpha}\, m\right) \tag{2.15}$$

$$f_1(v) = \frac{1}{\sigma}\left(\frac{1-\beta}{\alpha}\, n + \frac{1-\beta}{\alpha^2}\, m\right) \tag{2.16}$$

it being understood that m, n, α, β and σ are velocity dependent and that $\tan^{-1}\alpha$ is restricted to its principal value. When these expressions for f_0 and f_1 are substituted in (2.11) and (2.12), the following simultaneous integral equations for m and n are obtained:

$$m(v) = \int_v^{V_0} dv'\, K^{(1)}(v,v')\, m(v') + \int_v^{V_0} dv'\, K^{(2)}(v,v')\, n(v') \tag{2.17}$$

$$m(v) = g(v) - \int_v^{v_0} dv' K^{(3)}(v,v') m(v') + \int_v^{v_0} dv' K^{(4)}(v,v') m(v') \tag{2.18}$$

where
$$K^{(1)}(v,v') = 3\bar{\mu}_0(v') \frac{1-\beta(v')}{\alpha^2(v')} \frac{\sigma_s(v,v')}{\sigma(v')}$$

$$K^{(2)}(v,v') = 3\bar{\mu}_0(v') \frac{1-\beta(v')}{\alpha(v')} \frac{\sigma_s(v,v')}{\sigma(v')}$$

$$K^{(3)}(v,v') = \frac{1-\beta(v')}{\alpha(v')} \frac{\sigma_s(v,v')}{\sigma(v')}$$

$$K^{(4)}(v,v') = \beta(v') \frac{\sigma_s(v,v')}{\sigma(v')}$$

$$g(v) = \nu \chi(v) \int_0^{v_0} dv' \frac{\sigma_f(v')}{\sigma(v')} \left[\beta(v') n(v') + \frac{1-\beta(v')}{\alpha(v')} m(v')\right]$$

$$= c\,\chi(v) \tag{2.19}$$

c being an undetermined constant.

In order to show that (2.17) and (2.18) actually define the functions n and m, we require the following theorem:[4]

The integral equation

$$m(v) = g(v) + \int_v^{v_0} dv' K(v,v') m(v') \tag{2.20}$$

where $|K(v,v')| < M$ for all v, v', has exactly one solution which is finite everywhere. This solution is given by

$$m(v) = g(v) + \int_v^{v_0} dv' N(v,v') g(v') \tag{2.21}$$

where

$$N(v,v') = \sum_{m=1}^{\infty} K_m(v,v')$$

is the Neumann series in the iterated kernels, the latter being de-

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[4] V. Volterra, Lecons sur les Équations Integrales, Paris, Gauthier - Villars, 1913, p. 50 - 52.

fined by

$$K_1(v,v') = K(v,v')$$

$$K_{m+1}(v,v') = \int_v^{v'} dv'' K(v,v'') K_m(v'',v') \qquad (2.22)$$

The solution of (2.17) and (2.18) can now be obtained by a procedure similar to that used for simultaneous algebraic equations. Using the theorem just proved and considering n(v) as given, equation (2.17) can be solved for m(v) in terms of n(v):

$$m(v) = \int_v^{v_0} dv' K^{(2)}(v,v') m(v') + \int_v^{v_0} dv' N^{(1)}(v,v') \int_v^{v_0} dv'' K^{(2)}(v',v'') m(v'') \qquad (2.23)$$

where $N^{(1)}(v,v')$ is the Neumann series corresponding to $K^{(1)}(v,v')$

Substituting this expression for m(v) in (2.18) yields an equation of the form

$$m(v) = g(v) + \int_v^{v_0} dv' K(v,v') m(v') \qquad (2.24)$$

containing n(v) only, where g(v) is the quantity defined by (2.19) and

$$K(v,v') = K^{(4)}(v,v') - \int_v^{v'} dv'' K^{(3)}(v,v'') K^{(2)}(v'',v') - \int_v^{v'} dv'' K^{(3)}(v,v'') \int_{v''}^{v'} dv''' N^{(1)}(v'',v''') K^{(2)}(v''',v') \qquad (2.25)$$

Equation (2.24) has the solution

$$m(v) = g(v) + \int_{v}^{V_0} N(v,v')\, g(v') \tag{2.26}$$

where $N(v,v')$ is the Neumann series in the kernel (2.25) and $m(v)$ can now be found from (2.23). That $m(v)$ and $n(v)$ as defined by (2.23) and (2.26) are a solution of the system (2.17) and (2.18) is evident since all operations used in deriving (2.23) and (2.26) are reversible. The pair of functions $m(v)$ and $n(v)$ determined by (2.17) and (2.18) are unique for a given $g(v)$ since a different pair would lead to the same equations (2.23) and (2.26).

In order to determine ν, the values of $n(v)$ and $m(v)$ so obtained are substituted in the equation

$$\nu \int_{0}^{V_0} dv\, \frac{\sigma_f(v)}{\sigma(v)} \left[\beta(v)\, m(v) + \frac{1-\beta(v)}{\alpha(v)}\, m(v)\right] = c \tag{2.27}$$

which defines c (cf. (2.19)). It is clear from the form of (2.23) and (2.26) that $n(v)$ and $m(v)$ will both contain c as a factor; it can therefore be cancelled from (2.27) leaving an equation in which all quantities are known except ν. The presence of the arbitrary constant c is caused by not specifying the absolute value of the flux, and corresponds to the fact that a critical reactor will maintain a steady state at any power level. Since the value of c has no effect on the functional form of $m(v)$ and $n(v)$ or the value of ν, it will be taken as unity, so that m and n are both real.

The complex solution of the Boltzmann equation can now be written, using (2.7) and (2.13), as

$$\mathcal{F}(x,v,\mu) = \frac{1}{2\sigma(v)} \frac{n(v) - i\mu m(v)}{1 + i\alpha(v)\mu} e^{iKx}$$

$$= \frac{1}{2\sigma} \frac{(n - \alpha\mu^2 m)\cos Kx + \mu(\alpha n + m)\sin Kx}{1 + \alpha^2\mu^2}$$

$$+ \frac{i}{2\sigma} \frac{(n - \alpha\mu^2 m)\sin Kx - \mu(\alpha n + m)\cos Kx}{1 + \alpha^2\mu^2} \qquad (2.28)$$

That this function, with m(v) and n(v) defined by the integral equations (2.17) and (2.18), is a solution, can be seen by substituting it in place of F in (2.3). On cancelling the exponential e^{iKx} and equating coefficients of like powers of μ, the identities (2.11) and (2.12) result. Both the real and imaginary parts of F are solutions of (2.3); however, only the real part satisfies the symmetry condition (2.4). Using the definitions (2.1), (2.2), (2.9) and (2.10), the integrals (2.14) and retaining only the real parts,

$$F(x,v,\mu) = \frac{(n - \alpha\mu^2 m)\cos Kx + \mu(\alpha n + m)\sin Kx}{2\sigma(1 + \alpha^2\mu^2)} \qquad (2.29)$$

$$F_0(x,v) = \frac{1}{\sigma}\left(\beta n - \frac{1-\beta}{\alpha} m\right)\cos Kx = f_0(v)\cos Kx \qquad (2.30)$$

$$F_1(x,v) = \frac{1}{\sigma}\left(\frac{1-\beta}{\alpha} n + \frac{1-\beta}{\alpha^2} m\right)\sin Kx = f_1(v)\sin Kx \qquad (2.31)$$

Conditions (2.5) and (2.6) can be met by choosing

$$K = \frac{\pi}{2a} \qquad (2.32)$$

thereby determining K in terms of the reactor dimensions.

Most of the previously defined quantities can now be interpreted. From (2.30) and (2.31), f_0 and f_1 are the energy dependent factors of the flux and current respectively. From (2.8), n(v)dv is, neglecting the space factor, the number of neutrons entering the velocity interval dv in unit time by being born there as fission neutrons or scattering into the interval from a higher velocity. Since a steady state exists, it is also

the rate at which neutrons are lost by absorption, scattering and leakage. Comparing (2.12) with (2.3), it can be seen that m(v) is essentially the distortion term due to anisotropic scattering; its effect ((2.25) and (2.26)) is to decrease the flux and increase the outward current. The quantity g(v) (2.19) is the fission source term. Finally, $\alpha = \kappa / \sigma$ is, using (2.32) and neglecting the factor $\pi/2$, the ratio of the total mean free path to the half-width of the reactor, a quantity assumed to be small.

In satisfying the condition that the flux vanish on the boundary, it would be sufficient to take the parameter κ equal to any odd multiple of $\pi/2a$, so that the solution does not appear to be determined completely. For the case of isotropic scattering in the laboratory system, however, we shall show that only (2.32) leads to a solution of the problem. By properly choosing the complex functions $f^{(j)}(v,\mu)$, any solution of (2.3), (2.4), (2.5) and (2.6) can be written as the real part of

$$\mathcal{F}(x,v,\mu) = \sum_{j=0}^{\infty} f^{(j)}(v,\mu)\, e^{i\kappa_j x} \tag{2.33}$$

where $$\kappa_j = \frac{\pi}{a}\left(j + \tfrac{1}{2}\right) \tag{2.34}$$

Substituting (2.33) in (2.3) and equating coefficients of $e^{i\kappa_j x}$ leads to an equation of the form of (2.8) for each $f^{(j)}$. We suppose $\mathcal{F}(x,v,\mu)$ is a particular solution of the problem with ν the corresponding number of neutrons per fission required for criticality, and consider any two distinct modes, say f' and f''.

Following the previous procedure with $\bar{\mu}_0 = 0$ leads to the equations, corresponding to (2.18) and (2.32),

$$n'(v) = \chi(v) + \int_v^{v_0} dv'\, K'(v,v')\, n'(v')$$

$$\frac{1}{\nu} = \int_0^v dv\, \frac{\sigma_f(v)}{\sigma(v)}\, \beta'(v)\, n'(v) \qquad (2.35)$$

$$n''(v) = \chi(v) + \int_v^{v_0} dv'\, K''(v,v')\, n''(v')$$

$$\frac{1}{\nu} = \int_0^v dv\, \frac{\sigma_f(v)}{\sigma(v)}\, \beta''(v)\, n''(v) \qquad (2.36)$$

where

$$K'(v,v') = \beta'(v')\, \frac{\sigma_s(v,v')}{\sigma(v')} \quad , \quad K''(v,v') = \beta''(v')\, \frac{\sigma_s(v,v')}{\sigma(v')}$$

The constants c' and c'' have both been taken as unity since only ν is of interest. Suppose $K'' > K'$. Then $\alpha'' = K''/\sigma > \alpha'$ and $\beta'' = \frac{\tan^{-1}\alpha''}{\alpha''} < \beta'$ for all v. The Neumann series in $K''(v,v')$ is accordingly less than the one in $K'(v,v')$ and since

$$n'(v) = \chi(v) + \int_v^{v_0} dv'\, N'(v,v')\, \chi(v')$$

$$n''(v) = \chi(v) + \int_v^{v_0} dv'\, N''(v,v')\, \chi(v')$$

by (2.21), it follows that $n''(v) < n'(v)$ for all v.

From (2.35) and (2.36),

$$\int_0^{v_0} dv\, \frac{\sigma_f(v)}{\sigma(v)} \left[\beta'(v)\, n'(v) - \beta''(v)\, n''(v)\right] = \frac{1}{\nu} - \frac{1}{\nu} > 0$$

In view of this contradiction, the initial assumption that the solution contained more than one mode cannot be true; if several solutions of the problem exist, they each consist of only a single mode. But the only such solution which is never

negative in the reactor is the fundamental ($j = o$ in (2.34)), which therefore constitutes the unique solution of the problem.

III. ISOTROPIC SCATTERING

In this section some of the previous results will be specialized to the case in which scattering is isotropic in the laboratory system, and will apply approximately to a reactor consisting of heavy elements where moderation is due almost entirely to inelastic scattering.

Since $\bar{\mu}_0$ and hence m(v) are zero, the flux and current are given by

$$F_0(x,v) = f_0(v)\cos Kx = \frac{\beta(v)}{\sigma(v)} n(v)\cos Kx \tag{3.1}$$

$$F_1(x,v) = f_1(v)\sin Kx = \frac{1-\beta(v)}{\sigma(v)\alpha(v)} n(v)\sin Kx \tag{3.2}$$

where n(v) is the solution of the single equation

$$n(v) = \chi(v) + \int_v^{v_0} dv'\beta(v')\frac{\sigma_s(v,v')}{\sigma(v')} n(v') \tag{3.3}$$

and ν is then determined from

$$\nu \int_0^{v_0} dv \frac{\sigma_f(v)}{\sigma(v)} \beta(v) n(v) = 1 \tag{3.4}$$

corresponding to equations (2.30), (2.31), (2.18) and (2.27) respectively.

It is convenient to introduce the following notation: The probability that a neutron which has left velocity v by any means (scattering, leakage or absorption) will be scattered to a velocity between v and v + dv, will be designated by a(v, v′) dv. Since this is the ratio of the rate at which neutrons leave v by scattering into the interval dv, to the rate at which they leave by any means,

$$a(v,v') = \frac{\sigma_s(v,v')\, f_o(v')\, \cos \kappa x}{n(v)\, \cos \kappa x} = \beta(v') \frac{\sigma_s(v,v')}{\sigma(v')} \qquad (3.5)$$

The last equality follows from (3.1), which can be written

$$\frac{f_o}{n} = \frac{\beta}{\sigma}$$

Similarly, $f(v)$ is defined as the probability that a neutron which has left velocity v has done so by causing fission, and is given by

$$f(v) = \frac{\sigma_f(v)\, f_o(v)\, \cos \kappa x}{n(v)\, \cos \kappa x} = \beta(v) \frac{\sigma_f(v)}{\sigma(v)} \qquad (3.6)$$

Using these definitions, (3.3) and (3.4) can be written

$$n(v) = \chi(v) + \int_v^{v_o} dv'\, a(v,v')\, n(v') \qquad (3.7)$$

$$\nu \int_0^{v_o} dv\, f(v)\, n(v) = 1 \qquad (3.8)$$

The Neumann series solution of (3.7) can be written

$$n(v) = \sum_{j=0}^{\infty} n_j(v) \qquad (3.9)$$

provided

$$n_0(v) = \chi(v)$$

$$n_{j+1}(v) = \int_v^{v} dv'\, a(v,v')\, n_j(v) \qquad (3.10)$$

as can be seen by comparing (3.9) and (3.10) with (2.21) and (2.22). From (3.10) the quantity $n_j(v)dv$ can be interpreted as the number of neutrons entering the interval dv which have been scattered j times since birth as fission neutrons. It is evident that this

method of solution is not well suited to numerical computation unless such heavy absorption or leakage is present that a neutron makes only a few collisions before being lost from the reactor. An alternative is to start with a function more nearly representing the final distribution than the fission spectrum and proceed by iteration. However, the kernel does not have a simple form and the integrations must be carried out numerically. It is therefore desirable to replace (3.7) by a set of simultaneous algebraic equations which, because of their forms, can be solved immediately. If the velocity range is divided into N intervals numbered in order of decreasing velocity, (3.7) and (3.8) can be written

$$n_i = \chi_i + \sum_{j=1}^{i-1} a_{ij} n_j \tag{3.11}$$

$$\nu \sum_{i=1}^{N} f_i n_i = 1 \tag{3.12}$$

where n_i is the number of neutrons leaving the i^{th} interval in unit time, f_i is the probability that a neutron leaving the i^{th} interval has done so by causing a fission,

χ_i is the fraction of fission neutrons born in the i^{th} interval, and

a_{ij} is the probability that a neutron leaving interval j has been scattered to interval i. Since a neutron which has been scattered into the same interval is considered not to have left that interval, $a_{ii} = 0$ and the upper limit on the summation in (3.11) is i-1. The n_j's can be found recursively, starting with the highest velocity:

$$
\begin{aligned}
n_1 &= \chi_1 \\
n_2 &= \chi_2 + a_{21} n_1 \\
n_3 &= \chi_3 + a_{31} n_1 + a_{32} n_2 \\
&\vdots
\end{aligned}
\tag{3.13}
$$

after which ν is found from (3.12)

Instead of using the continuity equation (3.7), one can work with the following adjoint equation:

$$p(v) = f(v) + \int_0^v dv' \, a(v', v) \, p(v') \tag{3.14}$$

which defines the function p(v) in terms of the known quantities f(v) and a(v′,v). In order to find a condition similar to (3.8) which will determine ν in terms of p(v), multiply (3.7) by p(v) and (3.14) by n(v) and integrate with respect to v from o to v_0:

$$\int_0^{v_0} dv \, p(v) \, n(v) = \int_0^{v_0} dv \, p(v) \, \chi(v) + \int_0^{v_0} dv \, p(v) \int_v^{v_0} dv' \, a(v, v') \, n(v')$$

$$\int_0^{v_0} dv \, p(v) \, n(v) = \int_0^{v_0} dv \, f(v) \, n(v) + \int_0^{v_0} dv \, n(v) \int_0^{v} dv' \, a(v, v') \, p(v')$$

The two double integrals are equal, as can be seen by reversing the order of integration in either one and interchanging v and v′. Subtracting one equation from the other, and making use of (3.8), we have

$$\nu \int_0^v dv \, p(v) \, \chi(v) = 1 \tag{3.15}$$

as the required condition.

One can therefore interpret p(v) as the probability, averaged over the reactor, that a neutron with velocity v will eventually result in a fission. Equation (3.14) is the statement that this probability is the sum of two parts: the pro-

bability of causing fission immediately on leaving v; and the probability of first being scattered to some lower velocity v', times the probability that a neutron at v' will eventually cause fission, summed over all v'. Under this interpretation both (3.14) and (3.15), which requires the neutron distribution to reproduce itself, could have been written immediately, but the explicit form of $a(v,v')$ would not have been apparent.

For numerical solution, (3.14) and (3.15) can be replaced by

$$p_i = f_i + \sum_{j=N}^{i+1} p_j a_{ji} \tag{3.16}$$

and

$$\nu \sum_{i=1}^{N} p_i \chi_i = 1 \tag{3.17}$$

The solution of (3.16) is

$$\begin{aligned} p_N &= f_N \\ p_{N-1} &= f_{N-1} + a_{N\,N-1}\, p_N \\ &\vdots \end{aligned} \tag{3.18}$$

Formulating the problem in this way has some advantage over the continuity equation, since the fission spectrum enters only after the p_i's are determined. This permits investigation of the effect on criticality of various assumptions for the fission spectrum without repeating the major part of the calculations.

The adjoint equation is closely related to the method of successive collisions of De Marcus and Nelson (ref. 2) which they have used to study the transmission of particles through matter. Their method consists of describing the neutron distribution by a state vector ψ whose component ψ_i is the number of neutrons in

the i th cell in phase space, each cell corresponding to a particular specification of position, velocity and direction, of which there are a finite number. In addition to these 'free states', 'trap states' are included, corresponding to processes which remove neutrons from the free states, and having the property that a neutron cannot leave them. A matrix operator A is introduced whose element A_{ij} is the probability that a neutron which has left state j will go to state i (which may be a trap state). After k transitions, the neutron distribution is

$$\psi^{(k)} = A^k \psi$$

The ultimate distribution is given by

$$\psi^* = A^* \psi \tag{3.19}$$

where

$$A^* = \lim_{k \to \infty} A^k$$

If this limit exists, the equation

$$A^* A = A^* \tag{3.20}$$

holds. Since A is known, A* can be determined from (3.20), and then used in (3.19) to find the ultimate distribution of neutrons among the trap states.

Although this method was set up for a non-reproducing medium, it can be used to find the critical size of a reactor by taking as the trap states capture leading to fission, using the fission spectrum for the initial distribution, and considering only one neutron generation. A generation is taken as the interval between birth as a fission neutron and death by absorption or leakage. Under the

conditions we have assumed, the application of this method is greatly simplified, since the space distribution is known in advance. This reduces the number of unknowns and makes it possible to solve the resulting simultaneous equations recursively.

We take N free states, each state corresponding to a particular velocity range and numbered from 1 to N in order of decreasing velocity. In addition, we take the N+1 st state as a trap state, to which neutrons causing fission are sent. Making use of its definition, the matrix A is then of the form

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{21} & 0 & 0 & \cdots & 0 & 0 \\ a_{31} & a_{32} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & & 0 & 0 \\ f_1 & f_2 & \cdots & & f_N & 1 \end{pmatrix} \qquad (3.21)$$

The sub-matrix consisting of the a_{ij}'s is triangular since neutrons are only scattered downward in energy. The last column has the form shown, since neutrons in the trap state cannot leave it, and the form of the last row follows from the previous definition of the f_i's. It should be noted that the sums of the elements in each column are generally less than one. If additional trap states were included, corresponding to other ways by which neutrons leave the reactor, it would be possible to account for all the neutrons involved. For our purpose, however, only the disposition of those neutrons causing fission is of interest.

Assume a finite probability of loss in each transition from a free state, and consider the probability of passing from an initial free state j to some final state after an infinite number of transi-

tions. This is clearly zero if the final state is a free state, and p_j (cf. the definition on p.23) if the final state is the fission trap state. The iterated matrix is therefore

$$A^* = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & 0 \\ p_1 & \cdots & p_N & 1 \end{pmatrix} \tag{3.22}$$

Substituting (3.21) and (3.22) in (3.20) and multiplying out leads to

$$p_i = f_i + \sum_{j=N}^{i+1} p_j a_{ji} \tag{3.23}$$

If we take as the initial distribution

$$\chi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \\ 0 \end{pmatrix}$$

the ultimate disposition of the neutrons of a particular generation will be given by the N+1 st component of the column vector $A^*\chi$ which is

$$\sum_{j=1}^{N} p_j \chi_j$$

In order for this generation to reproduce itself, it is necessary that

$$\nu \sum_{j=1}^{N} p_j \chi_j = 1 \tag{3.24}$$

Comparison of (3.23) and (3.24) with (3.16) and (3.17) shows that formulating the problem in terms of the finite difference approxi-

mation to the adjoint equation (3.14) leads to the same set of equations as the method of Nelson and De Marcus.

IV. MODERATION BY HYDROGEN

In this section the general treatment of Section II will be applied to a reactor containing hydrogen and an element which scatters isotropically and without energy loss[1]. It is convenient to use the neutron energy as an independent variable instead of the velocity, since the distribution in energy of neutrons scattered by hydrogen is uniform and depends only on the initial energy[2].

If $n(E)dE$ is re-defined as the rate at which neutrons leave the energy interval dE by any means, and similar definitions are made for the other quantities, equations (2.17) and (2.18) become

$$n(E) = \int_E^{E_0} dE' \, K^{(1)}(E') \, n(E') + \int_E^{E_0} dE' \, K^{(2)}(E') \, n(E') \qquad (4.1)$$

$$n(E) = \chi(E) - \int_E^{E_0} dE' \, K^{(3)}(E') \, n(E') + \int_E^{E_0} dE' \, K^{(4)}(E') \, n(E') \qquad (4.2)$$

where E_0 is the energy corresponding to v_0. Since $\mu_0 = 2/3$ for hydrogen[3], the kernels are

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[1] Similar results have been obtained by J. Ashkin at Los Alamos under these assumptions, and with a more general scattering law for hydrogen. I am indebted to T. A. Welton for bringing this to my attention.

[2] H. Soodak and E. C. Campbell, Elementary Pile Theory, AECD-2201, Oak Ridge, Technical Information Branch, Atomic Energy Commission, May 1949, p. 5.

[3] Ibid, p. 6

$$K^{(1)}(E) = 2\,\frac{1-\beta}{\alpha^2}\,\frac{\sigma_{sH}}{\sigma}\,\frac{1}{E}$$

$$K^{(2)}(E) = 2\,\frac{1-\beta}{\alpha}\,\frac{\sigma_{sH}}{\sigma}\,\frac{1}{E} \qquad (4.3)$$

$$K^{(3)}(E) = \frac{1-\beta}{\alpha}\,\frac{\sigma_{sH}}{\sigma}\,\frac{1}{E}$$

$$K^{(4)}(E) = \beta\,\frac{\sigma_{sH}}{\sigma}\,\frac{1}{E}$$

where σ_{sH} is the scattering cross-section of the hydrogen present, since only collisions with hydrogen result in loss of energy. Differentiating (4.1) and (4.2) with respect to E yields the following simultaneous differential equations:

$$\frac{dm}{dE} = -K^{(1)}(E)\,m(E) - K^{(2)}(E)\,n(E)$$

$$\frac{dn}{dE} = \frac{d\chi}{dE} + K^{(3)}(E)\,m(E) - K^{(4)}(E)\,n(E) \qquad (4.4)$$

provided $\chi(E)$ is differentiable, with the boundary conditions

$$m(E_0) = 0$$

$$n(E_0) = \chi(E_0) \qquad (4.5)$$

obtained from (4.1) and (4.2) by letting E approach E_0.

Instead of using this general source term, however, we will solve (4.1) and (4.2) for the case

$$\chi(E) = \delta(E - E_0) \qquad (4.6)$$

where $\delta(E-E_0)$ is the Dirac delta function, corresponding to mono-energetic fission neutrons.

Let

$$m(E) = \bar{m}(E) + \delta(E - E_0) \tag{4.7}$$

where $\bar{n}(E)$ is a bounded function. Substituting this together with (4.6) in (4.1) and (4.2),

$$\begin{aligned} m(E) &= \int_E^{E_0} dE' \, K^{(1)}(E') \, m(E') + \int_E^{E_0} dE' \, K^{(2)}(E') \, \bar{m}(E') + K^{(2)}(E_0) \\ \bar{m}(E) &= -\int_E^{E_0} dE' \, K^{(3)}(E') \, m(E') + \int_E^{E_0} dE' \, K^{(2)}(E') \bar{m}(E') \\ &\quad + K^{(4)}(E_0) \end{aligned} \tag{4.8}$$

Letting E approach E_0, the boundary conditions on m and n are

$$\begin{aligned} m(E_0) &= K^{(2)}(E_0) \\ \bar{m}(E_0) &= K^{(4)}(E_0) \end{aligned} \tag{4.9}$$

In order to obtain an explicit solution we assume constant cross-section, so that using (4.3) the kernels can be written

$$K^{(i)}(E) = \frac{c_i}{E} \quad , \quad i = 1, 2, 3, 4 \tag{4.10}$$

Differentiating (4.8) leads to

$$\begin{aligned} \frac{dm}{dE} &= -\frac{c_1}{E} m(E) - \frac{c_2}{E} \bar{m}(E) \\ \frac{d\bar{m}}{dE} &= \frac{c_3}{E} m(E) - \frac{c_4}{E} \bar{m}(E) \end{aligned} \tag{4.11}$$

We assume solutions of the form

$$\begin{aligned} m(E) &= a \left(\frac{E}{E_0}\right)^{\rho} \\ \bar{m}(E) &= b \left(\frac{E}{E_0}\right)^{\rho} \end{aligned} \tag{4.12}$$

where the constants a, b and p are to be determined. Substituting in (4.11) leads to the equation

$$\begin{aligned} (\rho + C_1)\, m + C_2\, \bar{n} &= 0 \\ -C_3\, m + (\rho + C_4)\, \bar{n} &= 0 \end{aligned} \tag{4.13}$$

The condition that these hold for all energies is

$$\begin{vmatrix} \rho + C_1 & C_2 \\ -C_3 & \rho + C_4 \end{vmatrix} = 0 \tag{4.14}$$

which determines the two constants

$$\rho_{1,2} = -\left(\frac{C_1 + C_4}{2}\right) \pm \sqrt{\left(\frac{C_1 + C_4}{2}\right)^2 + (C_2 C_3 - C_1 C_4)} \tag{4.15}$$

The general solution of (4.11) is

$$\begin{aligned} m(E) &= a_1 \left(\frac{E}{E_0}\right)^{\rho_1} + a_2 \left(\frac{E}{E_0}\right)^{\rho_2} \\ \bar{n}(E) &= b_1 \left(\frac{E}{E_0}\right)^{\rho_1} + b_2 \left(\frac{E}{E_0}\right)^{\rho_2} \end{aligned} \tag{4.16}$$

Using the boundary conditions (4.9) we get

$$\begin{aligned} a_1 + a_2 &= \frac{C_2}{E_0} \\ b_1 + b_2 &= \frac{C_4}{E_0} \end{aligned} \tag{4.17}$$

Substituting (4.16) in (4.11) and using (4.9),

$$\begin{aligned} \frac{\rho_1}{E_0} a_1 + \frac{\rho_2}{E_0} a_2 &= -\frac{C_1}{E_0}\frac{C_2}{E_0} - \frac{C_2}{E_0}\frac{C_4}{E_0} \\ \frac{\rho_1}{E_0} b_1 + \frac{\rho_2}{E_0} b_2 &= \frac{C_3}{E_0}\frac{C_2}{E_0} - \frac{C_4}{E_0}\frac{C_4}{E_0} \end{aligned} \tag{4.18}$$

Solving (4.17) and (4.18) for a_1, a_2, b_1 and b_2, the solution to (4.1) and (4.2) is

$$m(E) = -\frac{c_1 c_2 + c_2 c_4 + c_2 p_2}{(p_1 - p_2) E_0} \left(\frac{E}{E_0}\right)^{p_1} + \frac{c_1 c_2 + c_2 c_4 + c_2 p_1}{(p_1 - p_2) E_0} \left(\frac{E}{E_0}\right)^{p_2}$$

$$m(E) = \frac{c_1 c_2 - c_4 c_4 - c_4 p_2}{(p_1 - p_2) E_0} \left(\frac{E}{E_0}\right)^{p_1} + \frac{-c_3 c_2 + c_4 c_4 + c_4 p_1}{(p_1 - p_2) E_0} \left(\frac{E}{E_0}\right)^{p_2}$$

Substituting these in (2.30) and (2.31) yields expressions for the flux and current.

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